Mathematica Moravica
Vol. **28**, No. **2** (2024), 1–16 doi: 10.5937/MatMor2402001R

Some coupled fixed point results for multi-valued nonlinear contractions in metric spaces using w-distance

Subhadip Roy, Parbati Saha,
Binayak S. Choudhury, Rajendra Pant*

Abstract. This study employs contractive criteria related to w-distances to design and address a non-linear multivalued fixed point problem. The results presented in this study are substantiated by the inclusion of three illustrative examples. In this paper, a dedicated portion is allocated to the examination of the use of w-distances. Specifically, it explores how the utilization of w-distances in the current context expands upon the findings derived from metric distances.

1. Introduction and preliminaries

Coupled fixed point results are a significant part of fixed point theory, which experienced substantial development with the publication of Bhaskar and Lakshmikantham's paper in 2006, where they introduced the coupled contraction mapping concept [3]. The concept of connected fixed points originates from the study undertaken by Guo et al. (1987) [11]. The significance of the subject matter remained largely unacknowledged until the aforementioned work authored by Bhaskar et al. was published. In the year 1969, Nadler [19] extended the field of metric fixed point theory to encompass set valued analysis. This was achieved through the publication of his set valued contraction mapping principle, which employed the concept of Hausdorff distance to measure the dissimilarity between closed and bounded subsets of a metric space. Several studies have examined coupled fixed point results related to single-valued mappings [5, 6, 8, 14, 20]. In contrast, other studies have focused on multi-valued mappings [1, 7, 23, 25].

2020 *Mathematics Subject Classification.* Primary: 47H10, 54H25.
Key words and phrases. Multi-valued contraction, w-distance, ceiling distance, coupled fixed point.
Full paper. Received 14 January 2024, accepted 7 July 2024, available online 21 August 2024.

Definition 1 ([19])**.** Consider a metric space $(\mathcal{Z}, \varrho)$ and the mapping $h : CB(\mathcal{Z}) \times CB(\mathcal{Z}) \to \mathbb{R}$ defined as

$$h(\mathcal{A}, \mathcal{B}) = \max\left\{\Delta(\mathcal{A}, \mathcal{B}), \Delta(\mathcal{B}, \mathcal{A})\right\},$$

where

$$\Delta(\mathcal{A}, \mathcal{B}) = \sup_{p\in\mathcal{A}} \inf_{q\in\mathcal{B}} \varrho(p, q).$$

Then h is a metric on $CB(\mathcal{Z})$, known as the Hausdorff metric induced by ϱ.

Definition 2 ([7])**.** Consider the mappings $\mathcal{Q} : \mathcal{Z} \to \mathcal{Z}$ and $F : \mathcal{Z} \times \mathcal{Z} \to CL(\mathcal{Z})$, where $(\mathcal{Z}, \varrho)$ is a metric space. Suppose there exists $(z_1, z_2) \in \mathcal{Z}\times\mathcal{Z}$ such that $\mathcal{Q}(z_1, z_2) = z_1$ and $\mathcal{Q}(z_2, z_1) = z_2$, then (z_1, z_2) is called a coupled fixed point of $\mathcal{Q}$, i.e., (z_1, z_2) is a coupled fixed point of F if $z_1 \in F(z_1, z_2)$ and $z_2 \in F(z_2, z_1)$.

As we already noted, there has been quite a good number of papers published on both single valued and multi-valued coupled fixed points. Several concepts of ordinary mappings could be extended to coupled mappings. This has yielded new results, especially in the field of fixed point theory. Also, various types of applicability of coupled mappings are commendable as is evidence through papers like [2, 12]. The concept of w-distances mentioned above was first introduced in 1996 by Kada et al. in their paper [13]. It has been realized through results published in the following years that the use of w-distance inequalities in fixed point theory of metric space is capable of making substantial advancement of the subject by extending some existing results and also by creating larger classes of function to which the results based on w-distance are applicable. The book [24] by Rakočević provides a comprehensive account of fixed point theory developed on the basis of w-distances. Some more recent references in fixed point theory where w-distance is used are noted in [4, 10, 16, 17, 21, 22, 27].

Definition 3 ([13])**.** Let $(\mathcal{Z}, \varrho)$ be a metric space. A w-distance is a mapping $\omega : \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ such that the following properties are fulfilled:

(w1) $\omega(\alpha, \gamma) \leq \omega(\alpha, \beta) + \omega(\beta, \gamma)$ for all $\alpha, \beta, \gamma \in \mathcal{Z}$;

(w2) $\omega(x, .)$ is lower semi-continuous (l.s.c.), i.e., if $\alpha \in \mathcal{Z}$ and $\beta_n \to \beta$ in $\mathcal{Z}$, then $\omega(\alpha, \beta) \leq \lim_{n\to\infty} \inf \omega(\alpha, \beta_n)$;

(w3) for each $\varepsilon > 0$ there exists $\delta > 0$ such $\omega(\gamma, \alpha) \leq \delta$ and $\omega(\gamma, \beta) \leq \delta$ imply $\varrho(\alpha, \beta) \leq \varepsilon$.

If ω is a w-distance defined on the metric space $(\mathcal{Z}, \varrho)$, then $(\mathcal{Z}, \varrho, \omega)$ will be termed as a w-distance space. If $(\mathcal{Z}, \varrho)$ is complete, then $(\mathcal{Z}, \varrho, \omega)$ will be called a complete w-distance space.

Lemma 1 ([13])**.** *Let $(\mathcal{Z}, \varrho, \omega)$ be a w-distance space.*

(i) *Let $\{a_n\}$ be a sequence in $\mathcal{Z}$ such that*

$$\lim_{n\to\infty} \omega(a_n, x) = \lim_{n\to\infty} \omega(a_n, y) = 0.$$

Then $x = y$. Particularly, if $\omega(z, x) = \omega(z, y) = 0$, then $x = y$.

(ii) *If $\omega(a_n, b_n) \leq u_n$ and $\omega(a_n, y) \leq v_n$ for any $n \in \mathbb{N}$, where $\{u_n\}$ and $\{v_n\}$ are sequences in $[0, \infty)$ converging to 0, then $\{b_n\}$ converges to y.*

Definition 4 ([26])**.** A w-distance ω defined over a metric space $(\mathcal{Z}, \varrho)$ is said to be a ceiling distance of ϱ if $\varrho(m, n) \leq \omega(m, n)$, for all $m, n \in \mathcal{Z}$.

Let $\mathcal{Z} = \mathbb{R}$ and ϱ be the usual distance function and $\omega(a, b) = \max\{m(b-a), n(a-b)\}$, where $m, n \geq 1$. Then ω is a ceiling distance of ϱ.

For any w-distance space $(\mathcal{Z}, \varrho, \omega)$ holds $\omega(a, \mathcal{M}) = \inf_{m\in\mathcal{M}} \omega(a, m)$. If ω is a ceiling distance of ϱ, then $\omega(a, \mathcal{M}) \leq \varrho(a, \mathcal{M})$. The following lemma was proved in [18]. It will be useful when we prove our main theorem in the following section.

Lemma 2 ([18])**.** *Let $(\mathcal{Z}, \varrho, \omega)$ be a w-distance space and $\mathcal{M} \in CL(\mathcal{Z})$. Let $a \in \mathcal{Z}$ be such that $\omega(a, a) = 0$. Then $\omega(a, \mathcal{M}) = 0$ if and only if $a \in \mathcal{M}$.*

For a w-distance space $(\mathcal{Z}, \varrho, \omega)$ and a mapping $\mathcal{Q} : \mathcal{Z} \times \mathcal{Z} \to CL(\mathcal{Z})$, we define:

$$f_w(x_1, x_2) := \omega(x_1, \mathcal{Q}(x_1, x_2)) + \omega(x_2, \mathcal{Q}(x_2, x_1)), \text{ for all } x_1, x_2 \in \mathcal{Z}.$$

Definition 5. Consider the mapping $\mathcal{Q} : \mathcal{Z} \times \mathcal{Z} \to CL(\mathcal{Z})$, where $(\mathcal{Z}, \varrho)$ is a metric space. For any $\mu_0, \nu_0 \in \mathcal{Z}$, $(\mu_n, \nu_n)_{n\in\mathbb{N}_0}$ is called a coupled orbit of $\mathcal{Q}$ if $\mu_{n+1} \in \mathcal{Q}(\mu_n, \nu_n)$ and $\nu_{n+1} \in \mathcal{Q}(\nu_n, \mu_n)$, for every $n \in \mathbb{N}_0$. A mapping $f : \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ is $\mathcal{Q}$-orbitally l.s.c. at (μ, ν) if for any coupled orbit $(\mu_n, \nu_n)_{n\in\mathbb{N}_0}$ of $\mathcal{Q}$ converging to (μ, ν) holds

$$f(\mu, \nu) \leq \liminf_{n\to\infty} f(\mu_n, \nu_n).$$

Let Φ denote the set of all functions $\varphi : [0, \infty) \to [c, 1)$, $0 < c < 1$, such that $\limsup_{r\to t^+} \varphi(r) < 1$, for all $t \in [0, \infty)$. In our main result we will use the class of functions Φ introduced in [29].

The aim of this study is to present multi-valued fixed point theorems for functions that meet specific inequalities in relation to w-distances. This will effectively extend certain findings achieved in metric spaces, where w-distances were not considered.

2. Main results

Theorem 1. *Let $(\mathcal{Z}, \varrho, \omega)$ be a complete w-distance space and $\mathcal{Q} : \mathcal{Z} \times \mathcal{Z} \to CL(\mathcal{Z})$ be a mapping. Assume $f_w : \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ defined by*

$$f_w(x_1, x_2) := \omega(x_1, \mathcal{Q}(x_1, x_2)) + \omega(x_2, \mathcal{Q}(x_2, x_1)), \text{ for all } x_1, x_2 \in \mathcal{Z}.$$

Suppose there exists $\varphi \in \Phi$ such that for each $x_1, x_2 \in \mathcal{Z}$ there exist $u_1 \in \mathcal{Q}(x_1, x_2)$ and $u_2 \in \mathcal{Q}(x_2, x_1)$ satisfying

$$\sqrt{\varphi(f_w(x_1, x_2))}[\omega(x_1, u_1) + \omega(x_2, u_2)] \leq f_w(x_1, x_2), \tag{1}$$

$$f_w(u_1, u_2) \leq \varphi(f_w(x_1, x_2))[\omega(x_1, u_1) + \omega(x_2, u_2)]. \tag{2}$$

Then

- (A1) *for every $\mu_0, \nu_0 \in \mathcal{Z}$ there exists a coupled orbit $(\mu_n, \nu_n)_{n \in \mathbb{N}_0}$ of $\mathcal{Q}$ such that $\lim_{n\to\infty} \mu_n = z_1$, $\lim_{n\to\infty} \nu_n = z_2$, for some $z_1, z_2 \in \mathcal{Z}$.*
- (A2) *f_w is $\mathcal{Q}$-orbitally l.s.c. at (z_1, z_2) if and only if $f_w(z_1, z_2) = 0$.*
- (A3) *(z_1, z_2) is a coupled fixed point of $\mathcal{Q}$, that is $z_1 \in \mathcal{Q}(z_1, z_2)$ and $z_2 \in \mathcal{Q}(z_2, z_1)$, provided $\omega(z_1, z_1) = \omega(z_2, z_2) = f_w(z_1, z_2) = 0$.*

Proof. Let μ_0, ν_0 be two arbitrary elements of $\mathcal{Z}$. Then, by assumptions (1) and (2), there exist $\mu_1 \in \mathcal{Q}(\mu_0, \nu_0)$ and $\nu_1 \in \mathcal{Q}(\nu_0, \mu_0)$ satisfying

$$\sqrt{\varphi(f_w(\mu_0, \nu_0))}[\omega(\mu_0, \mu_1) + \omega(\nu_0, \nu_1)] \leq f_w(\mu_0, \nu_0), \tag{3}$$

$$f_w(\mu_1, \nu_1) \leq \varphi(f_w(\mu_0, \nu_0))[\omega(\mu_0, \mu_1) + \omega(\nu_0, \nu_1)]. \tag{4}$$

From (3) and (4) we get

$$f_w(\mu_1, \nu_1) \leq \sqrt{\varphi(f_w(\mu_0, \nu_0))} f_w(\mu_0, \nu_0).$$

Again, by assumptions (1) and (2), we can choose $\mu_2 \in \mathcal{Q}(\mu_1, \nu_1)$ and $\nu_2 \in \mathcal{Q}(\nu_1, \mu_1)$ satisfying

$$\sqrt{\varphi(f_w(\mu_1, \nu_1))}[\omega(\mu_1, \mu_2) + \omega(\nu_1, \nu_2)] \leq f_w(\mu_1, \nu_1), \tag{5}$$

$$f_w(\mu_2, \nu_2) \leq \varphi(f_w(\mu_1, \nu_1))[\omega(\mu_1, \mu_2) + \omega(\nu_1, \nu_2)]. \tag{6}$$

From (5) and (6) we get

$$f_w(\mu_2, \nu_2) \leq \sqrt{\varphi(f_w(\mu_1, \nu_1))} f_w(\mu_1, \nu_1).$$

By repeating the process, we obtain a coupled orbit $(\mu_n, \nu_n)_{n \in \mathbb{N}_0}$ of $\mathcal{Q}$ satisfying

$$f_w(\mu_{n+1}, \nu_{n+1}) \leq \sqrt{\varphi(f_w(\mu_n, \nu_n))} f_w(\mu_n, \nu_n), \tag{7}$$

$$\sqrt{\varphi(f_w(\mu_n, \nu_n))}[\omega(\mu_n, \mu_{n+1}) + \omega(\nu_n, \nu_{n+1})] \leq f_w(\mu_n, \nu_n). \tag{8}$$

Since $\varphi(t) < 1$, from (7), we get that $\{f_w(\mu_n, \nu_n)\}$ is a strictly decreasing sequence. Therefore, $\lim_{n\to\infty} f_w(\mu_n, \nu_n) = r$ for some $r \geq 0$. Now, if $r > 0$, from (7) we have

$$\begin{aligned} r \leq \limsup_{n\to\infty} f_w(\mu_n, \nu_n) &\leq \limsup_{n\to\infty} \left(\sqrt{\varphi(f_w(\mu_n, \nu_n))} f_w(\mu_n, \nu_n)\right) \\ &\leq \limsup_{n\to\infty} \left(\sqrt{\varphi(f_w(\mu_n, \nu_n))}\right) \left(\limsup_{n\to\infty} f_w(\mu_n, \nu_n)\right) \\ &= \limsup_{n\to\infty} \left(\sqrt{\varphi(f_w(\mu_n, \nu_n))}\right) r \\ &< r. \end{aligned}$$

which is a contradiction. Therefore

$$\lim_{n\to\infty} f_w(\mu_n, \nu_n) = 0. \tag{9}$$

Suppose $\alpha = \limsup_{n\to\infty} \sqrt{\varphi(f_w(\mu_n, \nu_n))}$. Then, by the property of $\varphi, \alpha < 1$. Consider q such that $\alpha < q < 1$. Then, there is $n_0 \in \mathbb{N}$ such that

$$\sqrt{\varphi(f_w(\mu_n, \nu_n))} < q, \text{ for all } n \geq n_0.$$

Then, from (7) we get

$$f_w(\mu_{n+1}, \nu_{n+1}) \leq q f_w(\mu_n, \nu_n), \text{ for all } n \geq n_0. \tag{10}$$

From (10) we have

$$f_w(\mu_{n+1}, \nu_{n+1}) \leq q^{n+1-n_0} f_w(\mu_{n_0}, \nu_{n_0}), \text{ for all } n \geq n_0. \tag{11}$$

As $\varphi(t) \geq a$ for every $t \geq 0$, $0 < a < 1$, from (8) and (11) we have

$$\omega(\mu_n, \mu_{n+1}) + \omega(\nu_n, \nu_{n+1}) \leq c q^n f_w(\mu_{n_0}, \nu_{n_0}), \text{ for all } n \geq n_0, \ c = \frac{1}{\sqrt{a} q^{n_0}}.$$

Therefore, for $m > n \geq n_0$ holds

$$\begin{aligned} \omega(\mu_n, \mu_m) + \omega(\nu_n, \nu_m) &\leq \sum_{k=n}^{m-1} \omega(\mu_k, \mu_{k+1}) + \omega(\nu_k, \nu_{k+1}) \\ &\leq c f_w(\mu_{n_0}, \nu_{n_0}) \sum_{k=n}^{m-1} q^k \\ &\leq c f_w(\mu_{n_0}, \nu_{n_0}) \frac{q^n}{1-q}. \end{aligned} \tag{12}$$

Suppose $\varepsilon > 0$ be any given number and $\delta > 0$ satisfies (w3). From (12), there exists $n_\delta > n_0$ such that

$$c f_w(\mu_{n_0}, \nu_{n_0}) \frac{q^{n_\delta}}{1-q} < \delta \tag{13}$$

From (12) and (13), for all $m > n > n_\delta$, we obtain

$$\omega(\mu_{n_\delta}, \mu_m) + \omega(\nu_{n_\delta}, \nu_m) < \delta \quad \text{and} \quad \omega(\mu_{n_\delta}, \mu_n) + \omega(\nu_{n_\delta}, \nu_n) < \delta.$$

Therefore, from (w3) we have $\varrho(\mu_n, \mu_m) + \varrho(\nu_n, \nu_m) \le \varepsilon$, for all $m > n > n_\delta$.

Hence, $\{\mu_n\}$ and $\{\nu_n\}$ are both Cauchy sequence. By completeness of $(\mathcal{Z}, \varrho)$, there is $z_1, z_2 \in \mathcal{Z}$ such that $\lim_{n\to\infty} \mu_n = z_1$ and $\lim_{n\to\infty} \nu_n = z_2$.

For the second part, if f_w is $\mathcal{Q}$-orbitally l.s.c. at (z_1, z_2), then from (9)

$$0 \le f_w(z_1, z_2) \le \liminf_{n\to\infty} f_w(\mu_n, \nu_n) = 0.$$

Therefore, $f_w(z_1, z_2) = 0$. Conversely, suppose $f_w(z_1, z_2) = 0$ for some $z_1, z_2 \in \mathcal{Z}$. Let $(\mu_n, \nu_n)_{n\in\mathbb{N}}$ be a coupled orbit converging to (z_1, z_2), then

$$f_w(z_1, z_2) = 0 \le \liminf_{n\to\infty} f_w(\mu_n, \nu_n).$$

Hence, f_w is $\mathcal{Q}$-orbitally l.s.c.

Thirdly, suppose $\omega(z_1, z_1) = \omega(z_2, z_2) = f_w(z_1, z_2) = 0$, then

$$\omega(z_1, \mathcal{Q}(z_1, z_2)) = \omega(z_2, \mathcal{Q}(z_2, z_1)) = 0.$$

Then, from Lemma 2 follows $z_1 \in \mathcal{Q}(z_1, z_2)$ and $z_2 \in \mathcal{Q}(z_2, z_1)$, that is, (z_1, z_2) is a coupled fixed point of $\mathcal{Q}$. □

Theorem 2. *Let $(\mathcal{Z}, \varrho, \omega)$ be a complete w-distance space and $\mathcal{Q} : \mathcal{Z} \times \mathcal{Z} \to CL(\mathcal{Z})$ be a multi-valued mapping. Suppose condition (1) and (2) of Theorem 1 holds and*

$$\inf\big\{\omega(x_1, u_1) + \omega(x_2, u_2) + \omega(x_1, \mathcal{Q}(x_1, x_2)) \\ + \omega(x_2, \mathcal{Q}(x_2, x_1)) : x_1, x_2 \in \mathcal{Z}\big\} > 0,$$

for $u_1, u_2 \in \mathcal{Z}$, with $u_1 \notin \mathcal{Q}(u_1, u_2)$ and $u_2 \notin \mathcal{Q}(u_2, u_1)$. Then $\mathcal{Q}$ admits a coupled fixed point.

Proof. Similar to the proof of Theorem 1, there exist a coupled orbit $(\mu_n, \nu_n)_{\mathbb{N}_0}$ converging to (z_1, z_2), and for all $m > n \ge n_0$

$$\omega(\mu_n, \mu_m) + \omega(\nu_n, \nu_m) \le c f_w(\mu_{n_0}, \nu_{n_0}) \frac{q^n}{1-q}.$$

Since $\omega(x, .)$ is l.s.c.,

$$\begin{aligned}\omega(\mu_n, z_1) + \omega(\nu_n, z_2) &\le \liminf_{n\to\infty} \omega(\mu_n, \mu_m) + \liminf_{n\to\infty} \omega(\nu_n, \nu_m) \\ &\le \liminf_{n\to\infty} \left(\omega(\mu_n, \mu_m) + \omega(\nu_n, \nu_m)\right) \\ &\le c f_w(\mu_{n_0}, \nu_{n_0}) \frac{q^n}{1-q}, \quad \text{for all } n \ge n_0.\end{aligned}$$

Also, holds

$$\begin{aligned}\omega(\mu_n, \mathcal{Q}(\mu_n, \nu_n)) + \omega(\nu_n, \mathcal{Q}(\nu_n, \mu_n)) &\le \omega(\mu_n, \mu_{n+1}) + \omega(\nu_n, \nu_{n+1}) \\ &\le c q^n f\left(\mu_{n_0}, \nu_{n_0}\right) \text{ for all } n \ge n_0.\end{aligned}$$

Now, if $z_1 \notin \mathcal{Q}(z_1, z_2)$ and $z_2 \notin \mathcal{Q}(z_2, z_1)$, we have

$$\begin{aligned}
0 &< \inf \{\omega(x, z_1) + \omega(y, z_2) + \omega(x, \mathcal{Q}(x, y)) + \omega(y, \mathcal{Q}(y, x)) : x, y \in \mathcal{Z}\} \\
&\leq \inf \{\omega(\mu_n, z_1) + \omega(\nu_n, z_2) + \omega(\mu_n, \mathcal{Q}(\mu_n, \nu_n)) + \omega(\nu_n, \mathcal{Q}(\nu_n, \mu_n)) : n \geq n_0\} \\
&\leq c f_w(\mu_{n_0}, \nu_{n_0}) \left\{ \frac{q^n}{1-q} + q^n : n \geq n_0 \right\} = 0,
\end{aligned}$$

which is a contradiction. Hence, (z_1, z_2) is a coupled fixed point of $\mathcal{Q}$. □

Theorem 3. *Let $(\mathcal{Z}, \varrho, \omega)$ be a complete w-distance space, $\mathcal{Q} : \mathcal{Z} \times \mathcal{Z} \to CL(\mathcal{Z})$ be a multi-valued mapping and ω be a ceiling distance of ϱ. Suppose f_w is $\mathcal{Q}$-orbitally l.s.c. and conditions* (1) *and* (2) *of Theorem 1 are fulfilled. Then $\mathcal{Q}$ admits a coupled fixed point.*

Proof. Let μ_0, ν_0 be two arbitrary elements of $\mathcal{Z}$. Applying Theorem 1 we can generate a coupled orbit $(\mu_n, \nu_n)_{n \in \mathbb{N}_0}$ converging to (z_1, z_2). Since f_w is $\mathcal{Q}$-orbitally l.s.c., we have

$$f_w(z_1, z_2) \leq \liminf_{n \to \infty} f_w(\mu_n, \nu_n) = 0.$$

Therefore,

$$\omega(z_1, \mathcal{Q}(z_1, z_2)) = \omega(z_2, \mathcal{Q}(z_2, z_1)) = 0.$$

Since ω is a ceiling distance of ϱ, we have

$$\varrho(z_1, \mathcal{Q}(z_1, z_2)) \leq \omega(z_1, \mathcal{Q}(z_1, z_2)),$$

$$\varrho(z_2, \mathcal{Q}(z_2, z_1)) \leq \omega(z_2, \mathcal{Q}(z_2, z_1)).$$

Therefore,

$$\varrho(z_1, \mathcal{Q}(z_1, z_2)) = \varrho(z_2, \mathcal{Q}(z_2, z_1)) = 0.$$

Hence, $z_1 \in \mathcal{Q}(z_1, z_2)$ and $z_2 \in \mathcal{Q}(z_2, z_1)$. □

Example 1. Let $(\mathcal{Z}, \varrho, \omega)$ be a complete w-distance space, where $\mathcal{Z} = [0, 1]$, $\omega(x_1, x_2) = x_2$, for all $x_1, x_2 \in \mathcal{Z}$, and ϱ be the usual distance. Let $\mathcal{Q} : \mathcal{Z} \times \mathcal{Z} \to CL(\mathcal{Z})$ be a mapping given by

$$\mathcal{Q}(x_1, x_2) = \begin{cases} \left\{ \frac{x_1^2}{2} \right\}, & \text{if } x_1 \neq \frac{1}{2}, \\ \left\{ \frac{3}{4} \right\}, & \text{if } x_1 = \frac{1}{2}; \end{cases}$$

and $\varphi : [0, +\infty) \to [\frac{1}{6}, 1)$ is defined by

$$\varphi(t) = \begin{cases} \max \left\{ \frac{t}{2}, \frac{3}{4} \right\}, & \text{if } t \in [0, \frac{3}{2}], \\ \frac{1}{4}, & \text{if } t > \frac{3}{2}. \end{cases}$$

Now, $f_w(x_1, x_2)$ given by

$$f_w(x,y) = \begin{cases} \frac{x_1^2+x_2^2}{2}, & \text{if } x_1, x_2 \neq \frac{1}{2}, \\ \frac{x_1^2}{2} + \frac{3}{4}, & \text{if } x_1 \neq \frac{1}{2}, x_2 = \frac{1}{2}, \\ \frac{x_2^2}{2} + \frac{3}{4}, & \text{if } x_2 \neq \frac{1}{2}, x_1 = \frac{1}{2}, \\ \frac{3}{2}, & \text{if } x_1 = x_2 = \frac{1}{2}; \end{cases}$$

is $\mathcal{Q}$-orbitally l.s.c. We take into account the following instances.

Case I.
If $x_1, x_2 \neq \frac{1}{2}$, then $u_1 \in \mathcal{Q}(x_1, x_2) = \left\{\frac{x_1^2}{2}\right\}$ and $u_2 \in \mathcal{Q}(x_2, x_1) = \left\{\frac{x_2^2}{2}\right\}$. Therefore,

$$\begin{aligned} f_w(u_1, u_2) = \frac{x_1^4 + x_2^4}{8} &\leq \max\left\{\frac{\frac{x_1^2+x_2^2}{2}}{2}, \frac{3}{4}\right\} \frac{x_1^2 + x_2^2}{2} \\ &= \varphi(f_w(x_1, x_2))[\omega(x_1, u_1) + \omega(x_2, u_2)]. \end{aligned}$$

Case II.
If $x_1 \neq \frac{1}{2}, x_2 = \frac{1}{2}$, then $u_1 \in \mathcal{Q}(x_1, x_2) = \left\{\frac{x_1^2}{2}\right\}$ and $u_2 = \frac{3}{4} \in \mathcal{Q}(x_2, x_1)$. Therefore,

$$\begin{aligned} f_w(u_1, u_2) = \frac{x_1^4}{8} + \frac{9}{32} &\leq \max\left\{\frac{1}{2}\left(\frac{x_1^2}{2} + \frac{3}{4}\right), \frac{1}{4}\right\} \left(\frac{x_1^2}{2} + \frac{3}{4}\right) \\ &= \varphi(f_w(x_1, x_2))[\omega(x_1, u_1) + \omega(x_2, u_2)]. \end{aligned}$$

Case III.
If $x_2 \neq \frac{1}{2}, x_1 = \frac{1}{2}$, then $u_2 \in \mathcal{Q}(x_2, x_1) = \left\{\frac{x_2^2}{2}\right\}$ and $u_1 = \frac{3}{4} \in \mathcal{Q}(x_1, x_2)$. Thus,

$$\begin{aligned} f_w(u_1, u_2) = \frac{x_2^4}{8} + \frac{9}{32} &\leq \max\left\{\frac{1}{2}\left(\frac{x_2^2}{2} + \frac{3}{4}\right), \frac{3}{4}\right\} \left(\frac{x_2^2}{2} + \frac{3}{4}\right) \\ &= \varphi(f_w(x_1, x_2))[\omega(x_1, u_1) + \omega(x_2, u_2)]. \end{aligned}$$

Case IV.
If $x_1, x_2 = \frac{1}{2}$, then $u_1 = \frac{3}{4} \in \mathcal{Q}(x_1, x_2) = \left\{\frac{3}{4}\right\}$ and $u_2 = \frac{3}{4} \in \mathcal{Q}(x_2, x_1) = \left\{\frac{3}{4}\right\}$. Then,

$$\begin{aligned} f_w(u_1, u_2) &= \omega(u_1, \mathcal{Q}(u_1, u_2)) + \omega(u_2, \mathcal{Q}(u_2, u_1)) \\ &= 2\omega\left(\frac{3}{4}, T\left(\frac{3}{4}, \frac{3}{4}\right)\right) = \frac{9}{16} \\ &\leq \varphi\left(\frac{1}{2}\right)\frac{3}{2} = \varphi(f_w(x_1, x_2))[\omega(x_1, u_1) + \omega(x_2, u_2)]. \end{aligned}$$

Put $(z_1, z_2) = (0,0)$ and $\mu_0 = \nu_0 = \frac{1}{2}$, $\mu_1 = \nu_1 = \frac{1}{4}$. Consider the coupled orbit $(\mu_n, \nu_n)_{\mathbb{N}_0}$ given by $\mu_n = \frac{1}{2}\mu_{n-1}^2$ and $\nu_n = \frac{1}{2}\nu_{n-1}^2$, for $n \geq 2$. Then converges to $(z_1, z_2) = (0,0)$. Also, $f_w(0,0) = \omega(0,0) = 0$. Consequently, every requirement of Theorem 1 is met. Thus, $\mathcal{Q}$ possesses a coupled fixed point which is $(0,0)$.

Example 2. Consider the complete w-distance space $(\mathcal{Z}, \varrho, \omega)$, where $\mathcal{Z} = \mathbb{R}^+, \omega(x_1, x_2) = x_2$, for all $x_1, x_2 \in \mathcal{Z}$, and ϱ be the usual distance. Let $\mathcal{Q} : \mathcal{Z} \times \mathcal{Z} \to CL(\mathcal{Z})$ be a mapping given by

$$\mathcal{Q}(x_1, x_2) = \begin{cases} \left\{\frac{x_1}{2}\right\} \cup [x_1 + 1, +\infty), & \text{if } x_1 \in [0,1), \\ \left\{\frac{5}{6}, x_1 - \frac{1}{4}\right\}, & \text{if } x_1 \geq 1; \end{cases}$$

and $\varphi : [0, +\infty) \to [\frac{1}{2}, 1)$ is defined by

$$\varphi(t) = \begin{cases} \frac{1}{2}, & \text{if } t \in [0, \frac{4}{3}], \\ \frac{3}{4}, & \text{if } t > \frac{4}{3}. \end{cases}$$

Now, $f_w(x_1, x_2)$ given by

$$f_w(x_1, x_2) = \begin{cases} \frac{x_1 + x_2}{2}, & \text{if } x_1, x_2 \in [0,1), \\ \frac{x_1}{2} + \min\left\{\frac{5}{6}, x_2 - \frac{1}{4}\right\}, & \text{if } x_1 \in [0,1), x_2 \geq 1, \\ \frac{x_2}{2} + \min\left\{\frac{5}{6}, x_1 - \frac{1}{4}\right\}, & \text{if } x_2 \in [0,1), x_1 \geq 1, \\ \min\left\{\frac{5}{6}, x_1 - \frac{1}{4}\right\} + \min\left\{\frac{5}{6}, x_2 - \frac{1}{4}\right\}, & \text{if } x_1, x_2 \geq 1; \end{cases}$$

is $\mathcal{Q}$-orbitally l.s.c. We consider the following cases.

Case I.
If $x_1, x_2 \in [0,1)$, then $u_1 \in \mathcal{Q}(x_1, x_2) = \left\{\frac{x_1}{2}\right\}$ and $u_2 \in \mathcal{Q}(x_2, x_1) = \left\{\frac{x_2}{2}\right\}$, thus holds

$$\begin{aligned} f_w(u_1, u_2) = \frac{x_1 + x_2}{4} &\leq \frac{1}{2}\left(\frac{x_1 + x_2}{2}\right) \\ &= \varphi(f_w(x_1, x_2))[\omega(x_1, u_1) + \omega(x_2, u_2)]. \end{aligned}$$

Case II.
If $x_1 \in [0,1)$ and $x_2 \geq 1$, then $u_1 = \left\{\frac{x_1}{2}\right\} \in \mathcal{Q}(x_1, x_2) = \left\{\frac{x_1}{2}\right\} \cup [x_1 + 1, +\infty)$ and $u_2 = \frac{5}{6} \in \mathcal{Q}(x_2, x_1)$, thus holds

$$\begin{aligned} f_w(u_1, u_2) = \frac{x_1}{4} + \frac{5}{12} &\leq \frac{1}{2}\left(\frac{x_1}{2} + \frac{5}{6}\right) \\ &= \varphi(f_w(x_1, x_2))[\omega(x_1, u_1) + \omega(x_2, u_2)]. \end{aligned}$$

Case III.
If $x_2 \in [0,1)$ and $x_1 \geq 1$, then $u_2 = \left\{\frac{x_2}{2}\right\} \in \mathcal{Q}(x_2, x_1) = \left\{\frac{x_2}{2}\right\} \cup [x_2 + 1, +\infty)$

and $u_1 = \frac{5}{6} \in \mathcal{Q}(x_1, x_2)$, thus holds

$$\begin{aligned} f_w(u_1, u_2) &= \frac{x_2}{4} + \frac{5}{12} \leq \frac{1}{2}\left(\frac{x_2}{2} + \frac{5}{6}\right) \\ &= \varphi(f_w(x_1, x_2))[\omega(x_1, u_1) + \omega(x_2, u_2)]. \end{aligned}$$

Case IV.
If $x_1, x_2 \geq 1$, then $u_1 = \frac{5}{6} \in \mathcal{Q}(x_1, x_2) = \left\{\frac{5}{6}, x_1 - \frac{1}{4}\right\}$ and $u_2 = \frac{5}{6} \in \mathcal{Q}(x_2, x_1) = \left\{\frac{5}{6}, x_2 - \frac{1}{4}\right\}$, thus holds

$$\begin{aligned} f_w(u_1, u_2) &= \omega(u_1, \mathcal{Q}(u_1, u_2)) + \omega(u_2, \mathcal{Q}(u_2, u_1)) \\ &= 2\omega\left(\frac{5}{6}, \mathcal{Q}\left(\frac{5}{6}, \frac{5}{6}\right)\right) = \frac{5}{6} \leq \varphi\left(\frac{1}{2}\right)\frac{5}{3} \\ &= \varphi(f_w(x_1, x_2))[\omega(x_1, u_1) + \omega(x_2, u_2)]. \end{aligned}$$

Also, $u_1 \notin \mathcal{Q}(u_1, u_2)$ and $u_2 \notin \mathcal{Q}(u_2, u_1)$ implies $(u_1, u_2) \neq (0, 0)$. Then

$$\begin{aligned} &\inf\big\{\omega(x_1, u_1) + \omega(x_2, u_2) + \omega(x_1, \mathcal{Q}(x_1, x_2)) \\ &\qquad + \omega(x_2, \mathcal{Q}(x_2, x_1)) : x_1, x_2 \in \mathcal{Z}\big\} \\ &= \inf\big\{u_1 + u_2 + \mathcal{Q}(x_1, x_2) + \mathcal{Q}(x_2, x_1) : x_1, x_2 \in \mathcal{Z}\big\} \\ &= u_1 + u_2 > 0. \end{aligned}$$

Therefore, all the conditions of Theorem 2 are satisfied. Hence, $\mathcal{Q}$ has a coupled fixed point.

Example 3. Let $\mathcal{Z} = [-1, 1]$ be equipped with the usual distance ϱ and $\omega(x_1, x_2) = |x_1| + |x_2|$. Then $(\mathcal{Z}, \varrho, \omega)$ is a complete w-distance space and ω is a ceiling distance of ϱ. Let $\mathcal{Q} : \mathcal{Z} \times \mathcal{Z} \to CL(\mathcal{Z})$ be a mapping given by

$$\mathcal{Q}(x_1, x_2) = \begin{cases} \left[-1, \frac{x_1}{4}\right], & \text{if } x_1 \in [-1, 0), \\ \left[0, \frac{x_1}{2}\right], & \text{if } x_1 \in [0, 1]; \end{cases}$$

and $\varphi : [0, +\infty) \to \left[\frac{1}{4}, 1\right)$ is defined by

$$\varphi(t) = \begin{cases} \frac{1}{4}, & \text{if } t \in [0, 3], \\ \frac{3}{4}, & \text{if } t > 3. \end{cases}$$

Then

$$f_w(x_1, x_2) = \begin{cases} |x_1| + |x_2| + \left|\frac{x_1}{4}\right| + \left|\frac{x_2}{4}\right|, & \text{if } x_1, x_2 \in [-1, 0), \\ |x_1| + |x_2| + \left|\frac{x_1}{4}\right|, & \text{if } x_1 \in [-1, 0), x_2 \in [0, 1], \\ |x_1| + |x_2| + \left|\frac{x_2}{4}\right|, & \text{if } x_1 \in [0, 1], x_2 \in [-1, 0), \\ |x_1| + |x_2|, & \text{if } x_1, x_2 \in [0, 1]; \end{cases}$$

is $\mathcal{Q}$-orbitally l.s.c. Now, if $x_1, x_2 \in [-1, 0)$, then $u_1 = \frac{x_1}{4} \in \mathcal{Q}(x_1, x_2) = \left[-1, \frac{x_1}{4}\right]$ and $u_2 = \frac{x_2}{4} \in \mathcal{Q}(x_2, x_1) = \left[-1, \frac{x_2}{4}\right]$, so we have

$$\begin{aligned} f_w(u_1,u_2) &= \omega(u_1,\mathcal{Q}(u_1,u_2)) + \omega(u_2,\mathcal{Q}(u_2,u_1)) \\ &= \left|\frac{x_1}{4}\right| + \left|\frac{x_2}{4}\right| + \left|\frac{x_1}{16}\right| + \left|\frac{x_1}{16}\right| \\ &\leq \frac{1}{4}\left[\omega(x_1,u_1)+\omega(x_2,u_2)\right] \\ &= \varphi(f_w(x_1,x_2))[\omega(x_1,u_1)+\omega(x_2,u_2)]. \end{aligned}$$

If $x_1 \in [-1,0)$ and $x_2 \in [0,1]$, take $u_1 = \frac{x_1}{4}$ and $u_2 = 0$. Then

$$\begin{aligned} f_w(u_1,u_2) &= \omega(u_1,\mathcal{Q}(u_1,0)) + \omega(0,\mathcal{Q}(0,u_1)) \\ &= \left|\frac{x_1}{4}\right| + \left|\frac{x_1}{16}\right| \leq \frac{1}{4}\left(|x_1| + \left|\frac{x_1}{4}\right| + |x_2|\right) \\ &= \frac{1}{4}\left[\omega(x_1,u_1)+\omega(x_2,u_2)\right] \\ &= \varphi(f_w(x_1,x_2))[\omega(x_1,u_1)+\omega(x_2,u_2)]. \end{aligned}$$

The remaining cases can be illustrated in a similar manner. Hence, by Theorem 3, $\mathcal{Q}$ has a coupled fixed point.

Note 1. In Theorem 1 by appropriately replacing the conditions (1) and (2), the following outcome is evident. Let $(\mathcal{Z},\varrho,\omega)$ be a complete w-distance space and $\mathcal{Q}:\mathcal{Z}\times\mathcal{Z}\to CL(\mathcal{Z})$ be a mapping. Assume $f_w:\mathcal{Z}\times\mathcal{Z}\to[0,\infty)$ defined by

$$f_w(x_1,x_2) := \omega(x_1,\mathcal{Q}(x_1,x_2)) + \omega(x_2,\mathcal{Q}(x_2,x_1)), \quad \text{for all } x_1,x_2 \in \mathcal{Z}.$$

Suppose there exists $\varphi \in \Phi$ such that for each $x_1,x_2 \in \mathcal{Z}$ there exist $u_1 \in \mathcal{Q}(x_1,x_2)$ and $u_2 \in \mathcal{Q}(x_2,x_1)$ satisfying

$$\sqrt{\varphi(\omega(x_1,u_1)+\omega(x_2,u_2))}[\omega(x_1,u_1)+\omega(x_2,u_2)] \leq f_w(x_1,x_2), \tag{14}$$

$$f_w(u_1,u_2) \leq \varphi(\omega(x_1,u_1)+\omega(x_2,u_2))[\omega(x_1,u_1)+\omega(x_2,u_2)]. \tag{15}$$

Then the following holds:

(A1) for each $\mu_0,\nu_0 \in \mathcal{Z}$ there exists a coupled orbit $(\mu_n,\nu_n)_{n\in\mathbb{N}_0}$ of $\mathcal{Q}$ such that $\lim\limits_{n\to\infty}\mu_n = z_1$, $\lim\limits_{n\to\infty}\nu_n = z_2$ for some $z_1,z_2 \in \mathcal{Z}$.

(A2) $f_w(z_1,z_2) = 0$ if and only if f_w is $\mathcal{Q}$-orbitally l.s.c. at (z_1,z_2).

(A3) (z_1,z_2) is a coupled fixed point of $\mathcal{Q}$, that is, $z_1 \in \mathcal{Q}(z_1,z_2)$ and $z_2 \in \mathcal{Q}(z_2,z_1)$, provided $\omega(z_1,z_1) = \omega(z_2,z_2) = f_w(z_1,z_2) = 0$.

In the above result (14) and (15) are respectively replacements of (1) and (2).

The proof of the above can be obtained by proceeding similarly as in the case of Theorem 1, due to which we omit the details of the proof.

3. DISCUSSIONS

The conditions used in Theorem 1 are formulated by borrowing the idea from Feng et al. [9]. Such types of condition are popularly known as Feing-Liu type contractions which have been discussed in several papers, like [15, 28, 29]

Samet and Vetro [25] established a result regarding coupled fixed points for multi-valued mappings within the framework of complete metric spaces. Their result in the complete metric spaces without order is the following.

Theorem 4. *Let $(\mathcal{Z}, \varrho)$ be a complete metric space and $F : \mathcal{Z}\times\mathcal{Z} \to CL(\mathcal{Z})$ be a mapping. Suppose the function $f_\varrho : \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ defined by*

$$f_\varrho(x_1, x_2) = \varrho(x_1, F(x_1, x_2)) + \varrho(x_2, F(x_2, x_1))$$

is l.s.c. If there exists $\varphi \in \Phi$ such that for all $x_1, x_2 \in \mathcal{Z}$ there exist $u_1 \in F(x_1, x_2)$ and $u_2 \in F(x_2, x_1)$ satisfying

$$\sqrt{\varphi(f_\varrho(x_1, x_2))}[\varrho(x_1, u_1) + \varrho(x_2, u_2)] \leq f_\varrho(x_1, x_2),$$

$$f_\varrho(u_1, u_2) \leq \varphi(f_\varrho(x_1, x_2))[\varrho(x_1, u_1) + \varrho(x_2, u_2)],$$

then F has a coupled fixed point.

Theorem 4 is a particular case covered by Theorem 1 when ω is assumed to be same as ϱ.

(a) In Example 1, take $x_1 = 0, x_2 = \frac{1}{2}$. Then $u_1 = 0 \in \mathcal{Q}(0, 1) = \{0\}$ and $u_2 = \frac{3}{4} \in \mathcal{Q}(1, 0) = \left\{\frac{3}{4}\right\}$ and

$$\begin{aligned}&\varrho(u_1, \mathcal{Q}(u_1, u_2))) + \varrho(u_2, \mathcal{Q}(u_2, u_1))\\ &= \varrho(0, 0) + \varrho\left(\frac{3}{4}, \left\{\frac{9}{32}\right\}\right) = \frac{15}{32}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\varrho(u_1, \mathcal{Q}(u_1, u_2))) + \varrho(u_2, \mathcal{Q}(u_2, u_1))\\ &\leq \varphi(f_\varrho(x_1, x_2))[\varrho(x_1, u_1) + \varrho(x_2, u_2)],\end{aligned}$$

so we have

$$\frac{15}{32} \leq \varphi(f_\varrho(x_1, x_2))\frac{1}{4}.$$

From the above relation we have $\varphi(f_d(x_1, x_2)) \geq \frac{15}{8} > 1$, which is a contradiction. Therefore, Theorem 4 can not be applied. It is evident that Theorem 1 includes Theorem 4.

(b) If we take $x_1 = x_2 = 1$ in the Example 2., then $u_1, u_2 \in \mathcal{Q}(1,1) = \left\{\frac{5}{6}, \frac{3}{4}\right\}$. For $u_1 = u_2 = \frac{5}{6}$, $\varrho(x_1, u_1) + \varrho(x_2, u_2) = \frac{1}{3}$, so

$$\begin{aligned}&\varrho(u_1, \mathcal{Q}(u_1, u_2))) + \varrho(u_2, \mathcal{Q}(u_2, u_1))\\ &= 2\varrho\left(\frac{5}{6}, \left\{\left\{\frac{5}{12}\right\} \cup \left[\frac{11}{6}, +\infty\right)\right\}\right) = \frac{5}{6}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\varrho(u_1, \mathcal{Q}(u_1, u_2))) + \varrho(u_2, \mathcal{Q}(u_2, u_1))\\ &\leq \varphi(f_\varrho(x_1, x_2))[\varrho(x_1, u_1) + \varrho(x_2, u_2)],\end{aligned}$$

giving

$$\frac{5}{6} \leq \varphi(f_\varrho(x_1, x_2))\frac{1}{3}.$$

Thus, from the above, we get $\varphi(f_\varrho(x_1, x_2)) \geq \frac{5}{2} > 1$, which is a contradiction.

For $u_1 = \frac{5}{6}$, $u_2 = \frac{3}{4}$, $\varrho(x_1, u_1) + \varrho(x_2, u_2) = \frac{5}{12}$, so

$$\begin{aligned}&\varrho(u_1, \mathcal{Q}(u_1, u_2))) + \varrho(u_2, \mathcal{Q}(u_2, u_1))\\ &= \varrho\left(\frac{5}{6}, \left\{\left\{\frac{5}{12}\right\} \cup \left[\frac{11}{6}, +\infty\right)\right\}\right)\\ &\quad + \varrho\left(\frac{3}{4}, \left\{\left\{\frac{3}{8}\right\} \cup \left[\frac{11}{8}, +\infty\right)\right\}\right) = \frac{19}{24}.\end{aligned}$$

Therefore, the following holds

$$\begin{aligned}&\varrho(u_1, \mathcal{Q}(u_1, u_2))) + \varrho(u_2, \mathcal{Q}(u_2, u_1))\\ &\leq \varphi(f_\varrho(x_1, x_2))[\varrho(x_1, u_1) + \varrho(x_2, u_2)],\end{aligned}$$

which yields

$$\frac{19}{24} \leq \varphi(f_\varrho(x_1, x_2))\frac{5}{12}.$$

From the above inequality we get $\varphi(f_\varrho(x_1, x_2)) \geq \frac{19}{10} > 1$, which is a contradiction. The conclusion remains contradictory when both $u_1 = u_2 = \frac{3}{4}$ and $u_1 = \frac{3}{4}, u_2 = \frac{5}{6}$ are considered, as $\varphi(f_\varrho(x_1, x_2)) > 1$. Therefore, Theorem 4 is not applicable.

Both of the above considerations show that the result in Theorem 4 is properly contained in Theorem 1. It can be seen from the above considerations that the introduction of w-distance as contributed in the present case by properly extending the previous result of Samet et al. in metric spaces. The examples 1, 2 also give us instances of new functions to which the coupled fixed point result established in the present context are applicable.

Acknowledgment

The suggestions of the referees are gratefully acknowledged.

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Subhadip Roy
Department of Mathematics
Indian Institute of Engineering Science and Technology
Shibpur, Howrah–711103
West Bengal
India
E-mail address: subhadip_123@yahoo.com

Parbati Saha
Department of Mathematics
Indian Institute of Engineering Science and Technology
Shibpur, Howrah–711103
West Bengal
India
E-mail address: parbati_saha@yahoo.co.in

Binayak S. Choudhury
Department of Mathematics
Indian Institute of Engineering Science and Technology
Shibpur, Howrah–711103
West Bengal
India
E-mail address: binayak12@yahoo.co.in

Rajendra Pant
Department of Mathematics & Applied Mathematics
University of Johannesburg
Auckland Park 2006
South Africa
E-mail address: pant.rajendra@gmail.com
rpant@uj.ac.za